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Prescribing the scalar curvature problem on the four-dimensional half sphere

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Abstract In this paper, we consider the problem of prescribing scalar curvature under minimal boundary conditions on the standard four-dimensional half sphere. We describe the lack of compactness of the associated variational problem and we give new existence and multiplicity results.

Mathematics Subject Classification 35J20 · 35J60 · 58J05

الملخص

في هذه الورقة نعتبر مسألة وصف التقوس في ظل شروط حدية دنيا على نصف الكرة القياسية رباعية الأبعاد. نصف نقص التراص لمسألة التغاير المرتبطة ونعطي نتائج جديدة حول الوجود والتعددية.

1 Introduction and main results

Let (M^n, g) be an n-dimensional Riemannian manifold with boundary, $n \geq 3$, and let $\tilde{g} = u^{4/(n-2)}g$ be a conformal metric to g, where u is a smooth positive function. Then, the scalar curvatures R_g and $R_{\tilde{g}}$ and the mean curvatures of the boundary h_g and $h_{\tilde{g}}$, with respect to g and $\tilde{g}$ respectively, are related by the following equations:

$$\begin{cases} -c_n \Delta_g u + R_g u = R_{\tilde{g}} u^{\frac{n+2}{n-2}} & \text{in } M^n \\ \frac{2}{n-2}\dfrac{\partial u}{\partial \nu} + h_g u \quad = h_{\tilde{g}} u^{\frac{n}{n-2}} & \text{on } \partial M^n \end{cases} \tag{1.1}$$

where $c_n = 4(n-1)/(n-2)$ and ν denotes the outward normal vector with respect to g.

In view of the above equations, the following problem naturally arises: given two functions $K : M^n \to \mathbb{R}$ and $H : \partial M^n \to \mathbb{R}$, does there exist a metric $\tilde{g}$ conformally equivalent to g, such that $R_{\tilde{g}} = K$ and $h_{\tilde{g}} = H$?

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According to Eq. (1.1), the answer to this question is equivalent to finding a smooth positive solution u of the following problem:

$$\begin{cases} -c_n \Delta_g u + R_g u = K u^{\frac{n+2}{n-2}} & \text{in } M^n, \\ \frac{2}{n-2}\frac{\partial u}{\partial \nu} + h_g u \quad = H u^{\frac{n}{n-2}} & \text{on } \partial M^n. \end{cases} \tag{1.2}$$

When K and H are constants, this problem is called "The Yamabe Problem on Manifolds with boundary". It has also been studied through the works [4,18,26,27,31,32]. When $K = 0$, the problem is called Boundary Mean curvature problem which has been studied by Escobar (see [28]) on manifolds which are not equivalent to the standard ball. On the ball, sufficient conditions in dimensions 3 and 4 are given in [1,2,25,29]. When $H = 0$, the problem is called scalar curvature under minimal boundary condition and has been studied in [14–17,23]. Previously, Cherrier [22] studied the regularity question for this equation. He showed that solutions of (1.2) which are of class H^1 are also smooth.

We observe that the above problem is a natural generalization of the well-known "Scalar Curvature Problems on Closed manifolds": to find a positive smooth solution to the following equation:

$$- c_n \Delta_g u + R_g u = K u^{\frac{n+2}{n-2}} \quad \text{in } M^n, \tag{1.3}$$

to which much work has been devoted (see [3,5–8,10,11,13,20,21,30,34,35,37]).
In this paper, we consider the case where $H = 0$, on the standard four-dimensional half sphere under minimal boundary conditions. More precisely, let K be a C^2 positive Morse function on S^4_+, we look for conditions on K to ensure the existence of a positive solution of the problem

$$\begin{cases} L_g u = -\Delta_g u + 2u = K u^3 & \text{in } S^4_+, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial S^4_+, \end{cases} \tag{1.4}$$

where g is the standard metric of $S^4_+ = \{x \in \mathbb{R}^5 / |x| = 1, x_5 > 0\}$.

The main analytic difficulty of this problem comes from the presence of the critical Sobolev exponent on the right hand side of our equation, which generates blow up and lack of compactness. Indeed, due to the fact that the embedding $H^1(S^4_+) \to L^4(S^4_+)$ is not compact, the Euler–Lagrange functional J associated with our problem fails to satisfy the Palais Smale condition. That is there exist noncompact sequences along which the functional is bounded and its gradient goes to zero. Therefore, it is not possible to apply the standard variational methods to prove the existence of solution. There are also topological obstructions of Kazdan–Warner type to solve (1.4) [similar to the one associated to (1.3)], and so a natural question arises: under which conditions on K, (1.4) has a positive solution?

This problem has been studied by Li [33], and Djadli-Malchiodi-Ould Ahmedou [24], on the three-dimensional standard half sphere, using the blow-up analysis of some subcritical approximations and the use of the topological degree tools. In [16,17], the authors gave some topological conditions on K to prescribe the scalar curvature under minimal boundary conditions on half spheres of dimension bigger than or equal to 4 using the method of "critical points at infinity" due to Bahri [9] and Bahri-Coron [11]. In particular, they obtained an Euler–Hopf-type criterium reminiscent to the formula obtained by Bahri-Coron [11] for the scalar curvature problem on S^3, see also Chang-Gursky-Yang [21].

In this paper, we give new existence as well as multiplicity results, extending the previous all known ones. To state our results, we need to introduce some notations and assumptions. We denote by G the Green's function of the conformal Laplacian L_g on S^4_+ and H its regular part defined by

$$\begin{cases} G(x, y) = (1 - \cos(d(x, y)))^{-1} + H(x, y), \\ \Delta H = 0 \text{ in } S^4_+, \quad \frac{\partial G}{\partial \nu} = 0 \text{ on } \partial S^4_+. \end{cases} \tag{1.5}$$

Let $0 < K \in C^2(S^4_+)$ be a positive Morse function. We say that the function K satisfies the condition $(\mathbf{H_0})$:

- If y is a critical point of K, then $-\frac{\Delta K(y)}{3K(y)} - 4H(y, y) \neq 0$,
- If y is a critical point of $K_1 = K_{|\partial S^4_+}$, then $\frac{\partial K}{\partial \nu}(y) < 0$.

Denoting $\mathcal{K}$ the set of critical point of K, we set

$$\mathcal{K}^+ := \left\{ y \in \mathcal{K} : \; -\frac{\Delta K(y)}{3K(y)} - 4H(y, y) > 0 \right\}.$$

To each p-tuple $\tau_p := (y_1, \ldots, y_p) \in \mathcal{K}^+$, we associate a matrix $M(\tau_p) = (M_{ij})$ defined by,

$$M_{ii} = -\frac{\Delta K(y_i)}{3K(y_i)^2} - 4\frac{H(y_i, y_i)}{K(y_i)}, \quad M_{ij} = -4\frac{G(y_i, y_j)}{\sqrt{K(y_i)K(y_j)}} \text{ for } i \neq j. \tag{1.6}$$

We denote by $\rho(\tau_p)$ the least eigenvalue of $M(\tau_p)$, and we say that a function K satisfies the condition $(\mathbf{H_1})$ if for every $\tau_p \in (\mathcal{K}^+)^p$, we have $\rho(\tau_p) \neq 0$. We set

$$\mathcal{F}_\infty := \left\{ \tau_p = (y_1, \ldots, y_p) \in (\mathcal{K}^+)^p / \; \rho(\tau_p) > 0 \right\}, \tag{1.7}$$

and we define an index $i : \mathcal{F}_\infty \to \mathbb{Z}$ defined by

$$i(\tau_p) = p - 1 + \sum_{i=1}^{p} (4 - \operatorname{ind}(K, y_i)),$$

where $\operatorname{ind}(K, y_i)$ denotes the Morse index of K at its critical point y_i.
Now, we state our main result.

Theorem 1.1 *Let $0 < K \in C^2(S^4_+)$ be a positive function satisfying the conditions $(\mathbf{H_0})$ and $(\mathbf{H_1})$. If there exists $k \in \mathbb{N}$, such that*

1. $$\sum_{\tau_p \in \mathcal{F}_\infty / \; i(\tau_p) \leq k} (-1)^{i(\tau_p)} \neq 1,$$

2. $$i(\tau_p) \neq k + 1, \;\; \forall \tau_p \in \mathcal{F}_\infty.$$

Then, there exists a solution to the problem (1.4) of Morse index less or equal than $k + 1$.

Moreover, for generic K, it holds

$$\sharp \mathcal{N}_{k+1} \geq \left| 1 - \sum_{\tau_p \in \mathcal{F}_\infty / \; i(\tau_p) \leq k} (-1)^{i(\tau_p)} \right|,$$

where $\mathcal{N}_{k+1}$ denotes the set of solutions of (1.4) having their Morse indices less than or equal to $k + 1$.

Please observe that, taking in the above k to be ℓ, where ℓ is the maximal index over all elements of $\mathcal{F}_\infty$, the second assumption is trivially satisfied. Therefore, in this case, we have the following corollary, which recovers the previous existence result of Ben Ayed et al. [17].

Corollary 1.2 *Let $0 < K \in C^2(S^4_+)$ be a positive function satisfying the conditions $(\mathbf{H_0})$ and $(\mathbf{H_1})$. If*

$$\sum_{\tau_p \in \mathcal{F}_\infty} (-1)^{i(\tau_p)} \neq 1,$$

then there exists at least one solution to (1.4).

Moreover, for generic K, it holds

$$\sharp \mathcal{S} \geq \left| 1 - \sum_{\tau_p \in \mathcal{F}_\infty} (-1)^{i(\tau_p)} \right|,$$

where $\mathcal{S}$ denotes the set of solutions of (1.4).

We point out the main new contribution of Theorem 1.1 is that we address here the case where the total sum in the above corollary equals 1, but a partial one is not equal 1. The main issue being the possibility to use such an information to prove the existence of solution to the problem (1.4). Moreover, our result does not only give existence results, but also, under generic conditions, gives a lower bound on the number of solutions of (1.4). Such a result is reminiscent to the celebrated Morse Theorem, which states that, the number of critical points of a Morse function defined on a compact manifold, is lower bounded in terms of the topology of the underlying manifold. Our result can be seen as some sort of Morse Inequality at Infinity. Indeed, it gives a lower bound on the number of metrics with prescribed curvature in terms of the topology at infinity.

The remainder of this paper is organized as follows. In Sect. 2, we set up the variational structure and the lack of compactness of Problem (1.4). In Sect. 3, we characterize the critical points at infinity associated with our problem. The last section is devoted to the proof of the main result.

2 Variational structure and lack of compactness

In this section, we recall the functional setting and the variational problem and its main features. Problem (1.4) has a variational structure, the Euler–Lagrange functional is

$$J(u) = \frac{\|u\|^2}{\left(\int_{S^4_+} K|u|^4\right)^{\frac{1}{2}}}.$$

The space $H^1(S^4_+)$ is equipped with the norm

$$\| u \|^2 = \int_{S^4_+} |\nabla u|^2 + 2u^2.$$

We denote by Σ the unit sphere of $H^1(S^4_+)$, and we set $\Sigma^+ = \{u \in \Sigma, u > 0\}$.

Problem (1.4) is equivalent to finding the critical points of J subjected to the constraint $u \in \Sigma^+$. The Palais–Smale condition fails to be satisfied for J on Σ^+. To describe the sequences failing the Palais–Smale condition, we need to introduce some notations. For $a \in S^4_+$ and $\lambda > 0$, let

$$\delta_{a,\lambda}(x) = \frac{\lambda}{\left(\lambda^2 + 1 + (1-\lambda^2)\cos d(a,x)\right)}, \tag{2.1}$$

where d is the geodesic distance on (S^4_+, g_0). This function satisfies

$$-\Delta_g \delta_{a,\lambda} + 2\delta_{a,\lambda} = 8\delta^3_{a,\lambda} \quad \text{in } S^4_+. \tag{2.2}$$

Let $P\delta_{a,\lambda}$ be the unique solution of

$$\begin{cases} -\Delta_g P\delta_{a,\lambda} + 2P\delta_{a,\lambda} = {\delta_{a,\lambda}}^3 & \text{in } S^4_+ \\ \frac{\partial P\delta_{a,\lambda}}{\partial \nu} = 0 & \text{on } \partial S^4_+. \end{cases}$$

We define now the set of potential critical points at infinity associated with the function J. Let, for $\varepsilon > 0$, $p \in \mathbb{N}^*$ and w either a solution of (1.4) or zero,

$$\begin{aligned} V(p,\varepsilon,w) = \Big\{ & u \in \Sigma^+ \text{ s.t. } \exists a_1,\ldots,a_p \in S^4_+, \exists \alpha_0,\alpha_1,\ldots,\alpha_p > 0 \quad \text{and} \quad \exists \lambda_1,\ldots,\lambda_p > \varepsilon^{-1} \\ & \text{with } \Big\| u - \alpha_0 w - \sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i} \Big\| < \varepsilon, \lambda_i d_i > \varepsilon^{-1}, |\alpha_0^2 J(u)^2 - 1| < \varepsilon \\ & \text{and } \left| \frac{\alpha_i^2 K(a_i)}{\alpha_j^2 K(a_j)} - 1 \right| < \varepsilon, \quad \varepsilon_{ij} < \varepsilon \; \forall\, i \neq j \in \{1,\ldots,p\} \Big\}, \end{aligned}$$

where $d_i = d(a_i, \partial S^4_+)$ and $\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2\right)^{-1}$.

The failure of Palais–Smale condition can be described, following the idea of [19,36,38] as follows:

Proposition 2.1 *Let (u_k) be a sequence in Σ^+, such that $J(u_k)$ is bounded and $\partial J(u_k)$ goes to zero. Then, there exists an integer $p \in \mathbb{N}^*$, a sequence $(\varepsilon_k) > 0$, ε_k tends to zero, and an extracted subsequence of u_k's, again denoted (u_k), such that $u_k \in V(p, \varepsilon_k, w)$, where w is zero or a solution of (1.4).*

If u is a function in $V(p, \varepsilon, w)$, one can find an optimal representation, following the ideas introduced in Proposition 5.2 of [9] (see also pages 348–350 of [10]). Namely, we have

Proposition 2.2 *For any $p \in \mathbb{N}^*$, there is $\varepsilon_p > 0$, such that if $\varepsilon \leq \varepsilon_p$ and $u \in V(p, \varepsilon, w)$, then the following minimization problem*

$$\min_{\substack{\alpha_i > 0,\, \lambda_i > 0,\, a_i \in S^4_+, \\ h \in T_w(W_u(w)).}} \left\| u - \sum_{i=1}^{p} \alpha_i P\delta_{(a_i,\lambda_i)} - \alpha_0(w+h) \right\|,$$

has a unique solution (α, λ, a, h), up to a permutation.

In particular, we can write u as follows:

$$u = \sum_{i=1}^{p} \alpha_i P\delta_{(a_i,\lambda_i)} + \alpha_0(w+h) + v,$$

where v belongs to $H^1(S^4_+) \cap T_w(W_s(w))$ and it satisfies (V_0), and $T_w(W_u(w))$ and $T_w(W_s(w))$ are the tangent spaces at w of the unstable and stable manifolds of w for a decreasing pseudo-gradient of J and (V_0) is the following:

$$(V_0): \begin{cases} \left\langle v, \psi \right\rangle = 0 \text{ for } \psi \in \left\{ P\delta_i, \dfrac{\partial P\delta_i}{\partial \lambda_i}, \dfrac{\partial P\delta_i}{\partial a_i}, i = 1, \ldots, p \right\} \\ \left\langle v, w \right\rangle = 0 \\ \left\langle v, h \right\rangle = 0 \text{ for all } h \in T_w W_u(w). \end{cases}$$

Here, $P\delta_i = P\delta_{(a_i,\lambda_i)}$ and $< ., . >$ denotes the scalar product defined on $H^1(S^4_+)$ by

$$< u, v >= \int_{S^4_+} \nabla u \nabla v + 2u\, v.$$

Notice that Proposition 2.2 is also true if we take $w = 0$, and therefore, $h = 0$. In the next, we will say that $v \in (V_0)$ if v satisfies (V_0).

Now, arguing as in [10, pages 326, 327 and 334], we have the following Morse lemma which completely gets rid of the v contributions and shows that it can be neglected with respect to the concentration phenomenon.

Proposition 2.3 *There is a $\mathcal{C}^1$ map to each $(\alpha_i, a_i, \lambda_i, h)$, such that $\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + \alpha_0(w+h)$ belongs to $V(p, \varepsilon, w)$ associates $\overline{v} = \overline{v}(\alpha, a, \lambda, h)$, such that $\overline{v}$ is unique and satisfies:*

$$J\left(\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + \alpha_0(w+h) + \overline{v}\right) = \min_{v \in (V_0)} \left\{ J\left(\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + \alpha_0(w+h) + v\right)\right\}.$$

Moreover,there exists a change of variables $v - \overline{v} \to V$, such that

$$J\left(\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + \alpha_0(w+h) + v\right) = J\left(\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + \alpha_0(w+h) + \overline{v}\right) + \| V \|^2 .$$

We notice that in the V variable, we define a pseudo-gradient by setting

$$\frac{\partial V}{\partial s} = -\mu V,$$

where μ is a very large constant. Then, at $s = 1$, $V(s) = e^{-\mu s} V(0)$, will be very small as we wish. This shows that, to define our deformation, we can work as if V was zero. The deformation will extend immediately with the same properties to a neighborhood of zero in the V variable.

3 Characterization of critical points at infinity

Following Bahri [9], we introduce the following definition.

Definition 3.1 A critical point at infinity of J in $\sum^+$ is a limit of a flow line $u(s)$ of the following equation:

$$\begin{cases} \frac{\partial u}{\partial s} = -\partial J(u) \\ u(0) = u_0 \end{cases}$$

such that $u(s)$ remains in $V(p, \varepsilon(s), w)$, for $s \geq s_0$.

Here, w is either zero or a solution of (1.4), and $\varepsilon(s)$ is some function tending to zero when $s \to +\infty$. Using Proposition 2.2, $u(s)$ can be written as:

$$u(s) = \sum_{i=1}^{p} \alpha_i(s) P\delta_{a_i(s),\lambda_i(s)} + \alpha_0(s)\big(w + h(s)\big) + v(s). \tag{3.1}$$

Denoting by $a_i = \lim a_i(s)$ and $\alpha_i = \lim \alpha_i(s)$, we denote by

$$(a_1, \ldots, a_p, w)_\infty \quad \text{or} \quad \sum_{i=1}^{p} \alpha_i P\delta_{a_i,\infty} + \alpha_0 w$$

such a critical point at infinity. If $w \neq 0$, it is called w-type.

3.1 Expansion of the gradient of the functional

Proposition 3.2 *For each $u = \sum_{j=1}^{p} \alpha_j P\delta_j + \alpha_0(w + h) \in V(p, \varepsilon, w)$, we have the following expansion:*

$$\Big\langle \partial J(u), h \Big\rangle \leq -c \parallel h \parallel^2 + O\left(\sum_{i=1}^{p} \frac{1}{(\lambda_i)^2}\right).$$

Proof We have

$$\Big\langle \partial J(u), h \Big\rangle = 2J(u)\left[\Big\langle u, h \Big\rangle - J(u)^2 \int_{S^4_+} K(x) u^3 h\right].$$

Observe that

$$\left(\sum_{j=1}^{p} \alpha_j P\delta_j + \alpha_0(w + h)\right)^3 = w^3 + 3w^2 h + O(|h|^3) + O\left(\sum_{j=1}^{p} (w + h)^2 P\delta_j + P\delta_j^3\right). \tag{3.2}$$

Thus,

$$\begin{aligned} \Big\langle \partial J(u), h \Big\rangle = 2J(u)\left[\alpha_0 \parallel h \parallel^2 - J(u)^2 \alpha_0^3 \left(\int_{S^4_+} w^3 h + 3w^2 h^2\right)\right] \\ + o(|h^2|_{L^\infty}) + O\left(\sum \int \delta_j^3 |h| + \delta_j |h|\right). \end{aligned} \tag{3.3}$$

Observe that

$$\int \delta_j^3 |h| \leq |h|_{L^\infty} \int \delta_j^3 = O\left(\frac{|h|_{L^\infty}}{\lambda_j}\right) = o(|h^2|) + O\left(\frac{1}{\lambda_j^2}\right), \tag{3.4}$$

$$\int \delta_j |h| = O\left(\frac{|h|_{L^\infty}}{\lambda_j}\right) = o(|h^2|). + O\left(\frac{1}{\lambda_j^2}\right), \tag{3.5}$$

Since the function h belongs to $T_w(W_u(w))$, which has a finite dimension equal to the index of w. Thus,

$$|h|_{L^\infty} = O(|h|_{H_0^1}), \quad \text{and} \quad \int K w^3 h =< w, h >= 0.$$

Therefore,

$$\Big\langle \partial J(u), h \Big\rangle = 2J(u)\Big[\alpha_0 \parallel h \parallel^2 - 3J(u)^2\alpha_0^3 \int_{S_+^4} w^2 h^2\Big] + o(|h^2|) + O\left(\sum_{i=1}^{p} \frac{1}{\lambda_j^2}\right). \tag{3.6}$$

Using the fact that $\alpha_0^2 J(u)^2 = 1 + o(1)$, we get,

$$\Big\langle \partial J(u), h \Big\rangle = c\Big[\parallel h \parallel^2 - 3\int_{S_+^4} w^2 h^2 \mathrm{d}x\Big] + O\left(\sum_{i=1}^{p} \frac{1}{(\lambda_i)^2}\right) + o\big(||h||^2\big).$$

Observe that arguing as in [10] (page 354), the quadratic form $Q_1(h,h) :=\parallel h \parallel^2 - 3\int_{S_+^4} K(x) w^2 h^2 \mathrm{d}x$ is negative definite. Hence, our proof follows. □

Proposition 3.3 *For each $u = \sum_{j=1}^{p} \alpha_j P\delta_j + \alpha_0(w+h) \in V(p,\varepsilon,w)$, we have the following expansion:*

$$\begin{aligned}\left\langle \partial J(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right\rangle = 2J(u)\Bigg[&\alpha_i c_1 \frac{\Delta K(a_i)}{K(a_i)\lambda_i^2} - c_2 \sum_{i\neq j} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \alpha_0 c_2 \frac{w(a_i)}{\lambda_i} \\ &-\alpha_i c_2 \frac{H(a_i,a_i)}{\lambda_i^2} - \sum_{i\neq j} \alpha_j c_2 \frac{H(a_i,a_j)}{\lambda_i \lambda_j} \\ &+ o\left(\sum_{i\neq j} \varepsilon_{ij} + ||h||^2 + o\left(\sum_{k=1}^{p} \frac{1}{\lambda_k^2}\right) + o\left(\sum_{k=1}^{p} \frac{1}{(\lambda_k d_k)^2}\right)\right)\Bigg],\end{aligned} \tag{3.7}$$

where c_1 and c_2 are some positive constants.

Proof We have

$$\left\langle \partial J(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right\rangle = 2J(u)\left[\left\langle u, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\right\rangle - J(u)^2 \int_{S_+^4} K(x) u^3 \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\right].$$

Using [16], for $u = \sum_{j=1}^{p} \alpha_j P\delta_j \in V(p,\varepsilon)$, we have

$$\begin{aligned}\left\langle \partial J(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right\rangle = 2J(u)\Bigg[&\alpha_i c_1 \frac{\Delta K(a_i)}{K(a_i)\lambda_i^2} - c_2 \sum_{i\neq j} \alpha_j \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \\ &-\alpha_i c_2 \frac{H(a_i,a_i)}{\lambda_i^2} - \sum_{i\neq j} \alpha_j c_2 \frac{H(a_i,a_j)}{\lambda_i \lambda_j} \\ &+ o\left(\sum_{i\neq j} \varepsilon_{ij} + \sum_{k=1}^{p} \frac{1}{\lambda_k^2} + \sum_{k=1}^{p} \frac{1}{(\lambda_k d_k)^2}\right)\Bigg].\end{aligned} \tag{3.8}$$

The stereographic projection and a direct calculation show that

$$\left\langle w, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right\rangle = -c_2 \frac{w(a_i)}{\lambda_i}(1+o(1)) + o\left(\frac{1}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^2}\right). \tag{3.9}$$

Similarly, we have

$$\left\langle h, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right\rangle = o\left(\frac{1}{\lambda_i^2}\right). \tag{3.10}$$

From another part, we have

$$\int_{S_+^4} K(x)u^3\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\mathrm{d}x = \int_{S_+^4} K(x)\big(\alpha_0 w\big)^3\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\mathrm{d}x + 3\int_{S_+^4} K(x)\big(\alpha_i P\delta_i\big)^2\big(\alpha_0 w\big)\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\mathrm{d}x$$
$$+2c_2 J(u)^{-2}\sum_{i\neq j}\alpha_j\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + o\left(\sum_{j\neq i}\varepsilon_{ij} + \sum_{k=1}^{p}(\frac{1}{\lambda_k^2} + \frac{1}{(\lambda_k d_k)^2}) + ||h||^2\right). \tag{3.11}$$

A straight for word computation yields:

$$\int_{S_+^4} K(x)\big(\alpha_0 w\big)^3\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\mathrm{d}x = -\alpha_0^3 c_2\frac{w(a_i)}{\lambda_i} + o\Big(\frac{1}{(\lambda_i d_i)^2} + \frac{1}{\lambda_i^2}\Big). \tag{3.12}$$

$$3\int_{S_+^4} K(x)\big(\alpha_i P\delta_i\big)^2\big(\alpha_0 w\big)\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\mathrm{d}x = -\alpha_0\alpha_i^2 c_2\frac{w(a_i)K(a_i)}{\lambda_i} + o\Big(\frac{1}{(\lambda_i d_i)^2} + \frac{1}{\lambda_i^2}\Big). \tag{3.13}$$

Using the above estimates and the fact that $J^2(u)\alpha_i^2 K(a_i) = 1 + o(1)$, Proposition 3.3 follows using similar argument as in [9]. □

Proposition 3.4 *For each* $u = \sum_{j=1}^{p}\alpha_j P\delta_j + \alpha_0(w+h) \in V(p,\varepsilon,w)$, *we have the following expansion:*

$$\Big\langle\partial J(u), \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\Big\rangle = -c_3 J(u)^3\alpha_i^3\frac{\nabla K(a_i)}{\lambda_i} + \frac{\alpha_i}{\lambda_i^3}c_2\frac{\partial H(a_i,a_i)}{\partial a_i}$$
$$-c_2\sum_{j\neq i}\alpha_j\left(\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i} - \frac{1}{\lambda_i^2\lambda_j}\frac{\partial H(a_i,a_j)}{\partial a_i}\right)(1+o(1)) + O\left(\sum_{i\neq j}\varepsilon_{ij}^2\lambda_j|a_i - a_j|\right)$$
$$+o\left(\frac{w(a_i)}{\lambda_i} + ||h||^2 + \sum_{j=1}^{p}\left(\frac{1}{\lambda_j^2} + \frac{1}{(\lambda_j d_j)^3}\right) + \sum_{k\neq j}\varepsilon kj^{\frac{3}{2}}\right). \tag{3.14}$$

Proof First observe that arguing as in [9], easy computations show the following estimates:

$$\Big\langle P\delta_i, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\Big\rangle = -\frac{c_2}{\lambda_i^3}\frac{\partial H(a_i,a_i)}{\partial a_i} + O\Big(\frac{1}{(\lambda_i d_i)^5}\Big), \tag{3.15}$$

$$\Big\langle P\delta_j, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\Big\rangle = c_2\left(\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i} - \frac{1}{\lambda_i^2\lambda_j}\frac{\partial H(a_i,a_j)}{\partial a_i}\right)$$
$$+O\left(\sum_{k=i,j}\frac{1}{(\lambda_k d_k)^5} + \varepsilon_{ij}^2\lambda_j|a_i - a_j|\right), \tag{3.16}$$

$$\int_{S_+^4} KP\delta_i^3\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i} = c_3\frac{\nabla K(a_i)}{\lambda_i}(1+o(1)) - \frac{c_2 K(a_i)}{\lambda_i^3}\frac{\partial H(a_i,a_i)}{\partial a_i} + O\Big(\frac{1}{(\lambda_i)^3}\Big) + O\Big(\frac{1}{(\lambda_i d_i)^5}\Big), \tag{3.17}$$

$$\int_{S_+^4} KP\delta_j^3\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i} = K(a_j)\Big\langle P\delta_j, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\Big\rangle$$
$$+O\left(\frac{1}{(\lambda_j d_j)^6} + \varepsilon_{ij}^2\log(\varepsilon_{ij}^{-1}) + \varepsilon_{ij}(\log(\varepsilon_{ij}^{-1}))\frac{1}{2}\frac{1}{\lambda_j}\right), \tag{3.18}$$

$$\int_{S_+^4} KP\delta_j \frac{1}{\lambda_i}\frac{\partial P\delta_i^3}{\partial a_i} = K(a_i)\Big\langle P\delta_j, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\Big\rangle$$
$$+O\left(\frac{1}{(\lambda_i d_i)^6} + \varepsilon_{ij}^2\log(\varepsilon_{ij}^{-1}) + \varepsilon_{ij}(\log(\varepsilon_{ij}^{-1}))^{\frac{1}{2}}\frac{1}{\lambda_i}\right), \tag{3.19}$$

$$\int_{S_+^4} Kw^3 \frac{1}{\lambda_i}\frac{\partial P\delta_i^3}{\partial a_i} = c_2\frac{\nabla w(a_i)}{\lambda_i^2} + o\Big(\frac{1}{\lambda_i^2}\Big) + O\Big(\frac{1}{(\lambda_i d_i)^4}\Big), \tag{3.20}$$

$$3\int_{S_+^4} KwP\delta_j^2 \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i} = c_2\frac{\nabla w(a_i)}{\lambda_i^2} + o\left(\frac{1}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^3} + \frac{\nabla w(a_i)}{\lambda_i^2}\right). \tag{3.21}$$

Using the above estimates and the fact that $J^2(u)\alpha_i^2 K(a_i) = 1 + o(1)$, Proposition 3.4 follows using similar argument as in [9]. □

3.2 Ruling out the existence of critical point at infinity in $V(p, \varepsilon, w)$ for $w \neq 0$

The aim of this section is to prove that, for K, a C^2 positive function satisfying the condition of theorem and w a solution of (1.4), then for each $p \in \mathbb{N}$, there is no critical point or critical point at infinity of J in the set $V(p, \varepsilon, w)$.

Proposition 3.5 *For $p \geq 1$, there exists a pseudo-gradient W, so that the following holds: There is a constant $c > 0$ independent of $u = \sum_{i=1}^p \alpha_i P\delta_i + \alpha_0(w + h) \in V(p, \varepsilon, w)$, so that*

1. $(\nabla J(u), W) \leq -c\left(\sum_{i=1}^p \left(\frac{1}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^3}\right) + \sum_{i\neq j}\varepsilon_{ij}^{\frac{3}{2}} + |h|^2\right)$
2. $(\nabla J(u + \bar{v}), \frac{\partial \bar{v}}{\partial(\alpha_i, a_i, \lambda_i)}(W)) \leq -c\left(\sum_{i=1}^p \left(\frac{1}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^3}\right) + \sum_{i\neq j}\varepsilon_{ij}^{\frac{3}{2}} + |h|^2\right).$

This pseudo-gradient satisfies the PS condition and it increases the least distance to the boundary along any flow line.

Proof Observe first, from Proposition 3.2, we have

$$\Big\langle \partial J(u), h\Big\rangle \leq -c \parallel h \parallel^2 + O\Big(\sum_{i=1}^p \frac{1}{\lambda_i^2}\Big). \tag{3.22}$$

Let $u = \sum_{i=1}^p \alpha_i P\delta_i + \alpha_0(w + h) \in V(p, \varepsilon, w)$. For $i \in \{1, \ldots, p\}$, we introduce the following condition:

$$\frac{1}{2^{p+1}}\sum_{i\neq k}\varepsilon_{ki} \leq \frac{H(a_i, a_j)}{\lambda_i\lambda_j}. \tag{3.23}$$

Let $d_0 > 0$ be a fixed small enough constant. We divide the set $\{1, \ldots, p\}$ into the following:
- $T_1 = \{1 \leq i \leq p/i$ satisfies (3.23) and $d_i < d_0\}$;
- $T_2 = \{1 \leq i \leq p/i$ does not satisfy (3.23) and $d_i < d_0\}$;
- $T_3 = \{1 \leq i \leq p/d_i \geq d_0\}$.

In $T_2 \cup T_3$, we order the λ_i's: $\lambda_{i_1} \leq \cdots \leq \lambda_{i_s}$. Let $c > 0$, a fixed constant small enough, we define

$$I := \{i_1\} \cup \{1 \leq j \leq s/c\lambda_k d_k \leq \lambda_{k-1}d_{k-1} \leq \lambda_k d_k, \forall k \leq j\}.$$

For a fixed constant $c' > 0$ small enough, we also define

$$I_{\lambda_{i_s}} := \{i_s\} \cup \{1 \leq k \leq s/c'\lambda_{i_{j+1}} \leq \lambda_{i_j} \leq \lambda_{i_{j+1}}, \forall j \geq k\}.$$

Let

$$Z_1 = -\sum_{k=1}^{s} 2^k \alpha_{i_k} \lambda_{i_k} \frac{\partial P\delta_{i_k}}{\partial \lambda_{i_k}}.$$

From Proposition 3.3, we have

$$(\nabla J(u), Z_1) \leq -c \sum_{i \in T_2 \cup T_3} \left(\frac{w(a_i)}{\lambda_i} + \sum_{k \neq i} \varepsilon_{ij} \right) + O\left(\sum_{i \in T_2 \cup T_3} \frac{1}{\lambda_i^2} \right)$$
$$+ o\left(\sum_{k=1}^{p} \frac{1}{(\lambda_k d_k)^2} + \sum_{k \neq i} \varepsilon_{ki}^{\frac{3}{2}} + \sum_{k=1}^{p} \frac{1}{\lambda_k^2} + |h|^2 \right) \tag{3.24}$$

Observe that if $i \in T_2 \cup T_3$, we have $w(a_i) > cd_i$, and therefore, $\frac{1}{\lambda_i^2} = o(\frac{w(a_i)}{\lambda_i})$. Thus,

$$(\nabla J(u), Z_1) \leq -c \sum_{i \in T_2 \cup T_3} \left(\frac{1}{\lambda_i^2} + \sum_{k \neq i} \varepsilon_{ij} \right) + o\left(\sum_{k=1}^{p} (\frac{1}{(\lambda_k d_k)^3} + \frac{1}{\lambda_k^2}) + \sum_{k \neq i} \varepsilon_{ki}^{\frac{3}{2}} + |h|^2 \right). \tag{3.25}$$

We need to add some more terms in our upper-bound. For this, let

$$X = \frac{1}{\lambda_{i_s}} \sum_{i \in I_{\lambda_{i_s}}} \frac{\partial P\delta_i}{\partial a_i} (-\alpha_i \nu_i).$$

Observe that $\frac{\partial H}{\partial \nu_i}(a_i, a_j) > 0$ or $\frac{\partial H}{\partial \nu_i}(a_i, a_j) = o\Big(d_i^{-1}(d_i d_j)^{-1}\Big)$. From Proposition 3.4, we derive that

$$(\nabla J(u), X) \leq -c\left(\frac{1}{\lambda_{i_s}} + \frac{1}{(\lambda_{i_s} d_{i_s})^3} \right) + \frac{1}{\lambda_{i_s}} O\Big(\sum_{k \neq j \in I_{\lambda_{i_s}}} \varepsilon_{kj}^2 \lambda_k \lambda_j |a_k - a_j||\nu_k - \nu_j| \Big)$$
$$+ \frac{1}{\lambda_{i_s}} O\Big(\sum_{k \in I_{\lambda_{i_s}}, j \notin I_{\lambda_{i_s}}} \varepsilon_{kj}^2 \lambda_k \lambda_j |a_k - a_j| \Big)$$
$$+ o\left(\sum_{i \neq j} \varepsilon_{ij}^{\frac{3}{2}} + \sum_{k=1}^{p} \frac{1}{(\lambda_k d_k)^3} + ||h||^2 \right) + O\left(\sum_{i \in I_{\lambda_{i_s}}} \frac{1}{\lambda_i^2} \right). \tag{3.26}$$

Since $i_s \in I$, we can make appearing the term $\frac{1}{(\lambda_{i_1} d_{i_1})^{-3}}$ in the last upper bound, and so all the terms $\frac{1}{(\lambda_i d_i)^{-3}}$. Observe that, if $k \in T_1$ and $k \neq j$, we have

$$\varepsilon_{ij}^{\frac{3}{2}} \leq c \sum_{i=1}^{p} \frac{1}{(\lambda_i d_i)^3}. \tag{3.27}$$

Thus, we can make appearing $\sum_{k \in T_1, j \neq k} \varepsilon_{ij}^{\frac{3}{2}}$ in the last upper bound. From another part, for $k, j \in I_{\lambda_{i_s}}$, we have $|\nu_k - \nu_j| = O(|a_k - a_j|)$. Thus,

$$\frac{1}{\lambda_{i_s}} \varepsilon_{kj}^2 \lambda_k \lambda_j |a_k - a_j||\nu_k - \nu_j| = \frac{1}{\lambda_{i_s}} O(\varepsilon_{kj}) = o\left(\frac{1}{\lambda_{i_s}} \right). \tag{3.28}$$

It remains to estimate the case where $k \in I_{\lambda_{i_s}}$ and $j \notin I_{\lambda_{i_s}}$. If $k \in T_2 \cup T_3$ or $j \in T_2 \cup T_3$, we have

$$\frac{1}{\lambda_{i_s}} \sum_{k \neq j \in I_{\lambda_{i_s}}} \varepsilon_{kj}^2 \lambda_k \lambda_j |a_k - a_j| = O(\lambda_j |a_k - a_j| \varepsilon_{kj}^2) = O\left(\left(\frac{\lambda_j}{\lambda_k} \right)^{\frac{1}{2}} \varepsilon_{kj}^{\frac{3}{2}} \right) = O(\varepsilon_{kj}). \tag{3.29}$$

If $j, k \in T_1$, we claim that:

$$\frac{1}{\lambda_{i_s}}\varepsilon_{kj}^2\lambda_k\lambda_j|a_k - a_j| = o\left(\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^3}\right). \tag{3.30}$$

In fact, if $j \in I$, we have $\lambda_j \le c'\lambda_k$. Then,

$$\frac{1}{\lambda_{i_s}}\varepsilon_{kj}^2\lambda_k\lambda_j|a_k - a_j| = O\left(\left(\frac{\lambda_j}{\lambda_k}\right)^{\frac{1}{2}}\varepsilon_{kj}\right). \tag{3.31}$$

From another part, since $j \in T_1$, we have

$$\varepsilon_{kj}^{\frac{3}{2}} \le c\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{\frac{3}{2}}(\lambda_j d_j)^{\frac{3}{2}}}. \tag{3.32}$$

Thus,

$$\frac{1}{\lambda_{i_s}}\varepsilon_{kj}^2\lambda_k\lambda_j|a_k - a_j| = o\left(\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^3}\right). \tag{3.33}$$

If $j \notin I$, let $c'' > 0$ be a fixed small enough constant. If $d_k \le c'' d_j$, then using the fact that $j \in T_1$, we get

$$\begin{aligned}\frac{1}{\lambda_{i_s}}\varepsilon_{kj}^2\lambda_k\lambda_j|a_k - a_j| &\le c\left(\frac{\lambda_j}{\lambda_k}\right)^{\frac{1}{2}}\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{\frac{3}{2}}(\lambda_j d_j)^{\frac{3}{2}}}\\ &\le cc''\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{\frac{3}{2}}(\lambda_j d_j)(\lambda_k d_k)^{\frac{1}{2}}}.\end{aligned} \tag{3.34}$$

Thus,

$$\frac{1}{\lambda_{i_s}}\varepsilon_{kj}^2\lambda_k\lambda_j|a_k - a_j| = o\left(\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^3}\right). \tag{3.35}$$

If $c'' d_j \le d_k \le \frac{1}{c''}d_j$, then

$$\begin{aligned}\frac{1}{\lambda_{i_s}}\varepsilon_{kj}^2\lambda_k\lambda_j|a_k - a_j| &\le \frac{c}{c''}\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{\frac{3}{2}}(\lambda_j d_j)(\lambda_k d_k)^{\frac{1}{2}}}\\ &\le \frac{c}{c''}\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{\frac{3}{2}}(\lambda_k d_k)^{\frac{3}{2}}}.\end{aligned} \tag{3.36}$$

Since $j \notin I$ and $k \in I$, we get the claim in this case. If $d_j \le c d_k$, we have $|a_k - a_j| \ge \frac{1}{2}d_k$, then $d_j \le 2c|a_k - a_j|$. We derive that

$$\begin{aligned}\frac{1}{\lambda_{i_s}}\varepsilon_{kj}^2\lambda_k\lambda_j|a_k - a_j| &\le c\frac{1}{(\lambda_j|a_k - a_j|)(\lambda_k|a_k - a_j|)^2}\\ &\le \frac{c}{c''}\sum_{i=1}^{p}\frac{1}{(\lambda_j d_j)(\lambda_k d_k)^2} = o\left(\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^3}\right),\end{aligned} \tag{3.37}$$

and the claim follows. Using now (3.30), we deduce

$$(\nabla J(u), X) \le -c\left(\frac{1}{\lambda_{i_s}} + \sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^3} + \sum_{k\in T_1, k\neq j}\varepsilon_{kj}^{\frac{3}{2}}\right) + O\left(\sum_{k\in T_2\cup T_3, k\neq j}\varepsilon_{kj}\right)$$
$$+o\left(\sum_{k\neq i}\varepsilon_{ij}^{\frac{3}{2}} + \sum_{i=1}^{p}\frac{1}{\lambda_i^2} + |h|^2\right). \tag{3.38}$$

We define now $Y = -\sum_{i\in T_1}\frac{1}{\lambda_i d_i}\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}$, we have

$$\left\langle \partial J(u), Y\right\rangle = 2J(u)\sum_{i\in T_1}\frac{1}{\lambda_i d_i}\Bigg[-\frac{w(a_i)}{\lambda_i} + O\Big(\frac{1}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^2} + \sum_{k\neq j}\varepsilon_{kj}\Big)$$
$$+o\Big(\sum_{i\neq j}\varepsilon_{ij}^{\frac{3}{2}} + ||h||^2 + \sum_{k=1}^{p}(\frac{1}{\lambda_k^2} + \frac{1}{(\lambda_k d_k)^3})\Big)\Bigg]. \tag{3.39}$$

Observe that $\frac{1}{\lambda_i d_i}\varepsilon_{kj} = O\Big(\frac{1}{(\lambda_i d_i)^3}\Big) + o(\varepsilon_{kj}^{\frac{3}{2}})$. Thus,

$$\left\langle \partial J(u), Y\right\rangle \le -c\left(\sum_{i\in T_1}\frac{1}{\lambda_i^2} + O\Big(\frac{1}{(\lambda_i d_i)^3}\Big)\right)$$
$$+o\left(\sum_{i\neq j}\varepsilon_{ij}^{\frac{3}{2}} + ||h||^2 + \sum_{k=1}^{p}\left(\frac{1}{\lambda_k^2} + \frac{1}{(\lambda_k d_k)^3}\right)\right). \tag{3.40}$$

For two fixed large enough constants $m_3 > m_2 > 0$, one has

$$\left\langle \partial J(u), Y + m_2 X + m_3 Z_1\right\rangle \le -c\left(\sum_{i\in T_1}\frac{1}{\lambda_i^2} + \sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^3} + \sum_{k\in T_1, k\neq j}\varepsilon_{kj}^{\frac{3}{2}}\right) + o\big(||h||^2\big). \tag{3.41}$$

The pseudo-gradient W will be defined by $W = m_4(Y + m_2 X + m_3 Z_1) + h$, where $m_4 > 0$ is a large enough fixed constant. Thus, the first claim of the proposition follows. The second claim can be obtained once we have (i) arguing as in [10]. □

Corollary 3.6 *Let K be a C^2 positive function satisfying the conditions* $(\mathbf{H_0})$ *and* $(\mathbf{H_1})$ *and let w be a non-degenerate critical point of J in Σ^+. Then, for each $p \ge 1$, there are no critical points or critical point at infinity in the set $V(p, \varepsilon, w)$.*

Now once mixed critical points at infinity are ruled out, it follows from [17], that the critical points at infinity are in one-to-one correspondence with the elements of the set $\mathcal{F}_\infty$ defined in (1.7), that is, a critical point at infinity corresponds to $\tau_p := (y_1, \ldots, y_p) \in (\mathcal{K}^+)^p$, such that the related matrix $M(\tau_p)$ defined in (1.6) is positive definite. Such a critical point at infinity will be denoted by τ_p^∞. Like a usual critical point, it is associated with a critical point at infinity x_∞ of the problem (1.4), which are combination of classical critical points with a one-dimensional asymptote, stable and unstable manifolds, $W_s^\infty(x_\infty)$ and $W_u^\infty(x_\infty)$. These manifolds can be easily described once a Morse-type reduction is performed, see [10]. In the following definition, we extend the notation of domination of critical points to critical points at infinity. Recall that $i(x_\infty)$, the Morse index, of such critical point at infinity is equal to the dimension of $W_u^\infty(y)_\infty$.

Definition 3.7 x_∞ is said to be dominated by another critical point at infinity x'_∞ if $W_u^\infty(x'_\infty)\cap W_s^\infty(x_\infty) \neq \emptyset$. If we assume that the intersection is transverse, then we obtain $i(x'_\infty) \ge (x_\infty) + 1$.

4 Proof of Theorem 1.1

Setting

$$\widehat{l} := \max\left\{ i(\tau_p),\ \tau_p \in \mathcal{F}_\infty \right\}. \tag{4.1}$$

For $k \in \left\{0, \ldots, \widehat{l}\right\}$, we define the following sets:

$$X_k^\infty = \bigcup_{\tau_p \in \mathcal{F}_\infty, i(\tau_p) \le k} \overline{W_u^\infty(\tau_p^\infty)},$$

where $\overline{W_u^\infty(\tau_p^\infty)}$ is the closure of the unstable manifold of (τ_p^∞), defined by adding to $W_u^\infty(\tau_p^\infty)$ the unstable manifolds of critical points or critical points at infinity dominated by (τ_p^∞). These manifolds are then of dimension $k-1$ at most, (See [10], Page 357). Therefore, X_k^∞ define a stratified set of top dimension $\le k$. Without loss of generality, we may assume that it is equal to k.

Now, for $\lambda > 0$, a fixed constant large enough and y_0 an absolute maximum of K on S_+^4, we define the following set:

$$\theta(X_k^\infty) = \left\{ \frac{tu + (1-t)\delta_{(y_0,\lambda)}}{\| tu + (1-t)\delta_{(y_0,\lambda)} \|},\ t \in [0,1],\ u \in X_k^\infty \right\}.$$

$\theta(X_k^\infty)$ define a contraction of X_k^∞ in Σ^+ of dimension $k+1$.

Now, we use the gradient flow of $(-\partial J)$ to deform $\theta(X_k^\infty)$. By transversally arguments, we can assume that the deformation avoids all critical points as well as critical points at infinity having their Morse index greater or equal to $k+2$. It follows then, by Proposition 7.24 and Theorem 8.2 of [12], that $\theta(X_k^\infty)$ retracts by deformation ($:\simeq$) on the set

$$X_k^\infty \bigcup \bigcup_{(\tau_p^\infty) < \theta(X_k^\infty), i(\tau_p^\infty) = k+1} W_u^\infty(\tau_p^\infty) \bigcup \bigcup_{\omega < \theta(X_k^\infty), i(\omega) \le k+1} W_u(\omega),$$

where (τ_p^∞) is a critical point at infinity of Morse index $k+1$ and dominated by $\theta(X_k^\infty)$, ω is a solution of (1.4) of Morse index $\le k+1$ and dominated by $\theta(X_k^\infty)$.

Now taking $\ell = k$, and using the assumption of Theorem 1.1, there are no critical points at infinity with index $k+1$, we derive that $\theta(X_k^\infty)$ retracts by deformation on the set

$$X_k^\infty \bigcup \bigcup_{\omega; \nabla J(\omega) = 0, \omega < \theta(X_k^\infty), i(\omega) \le k+1} W_u(\omega). \tag{4.2}$$

Now observe that, it follows from the above deformation retract that the problem (1.4) has necessary a solution ω with $i(\omega) \le k+1$. Otherwise it follows from (4.2) that

$$\theta(X_k^\infty) \simeq X_k^\infty. \tag{4.3}$$

It is easy to see that $\chi(\theta(X_k^\infty)) = 1$, since $\theta(X_k^\infty)$ is a contractible set. Thus, derive from (4.3), taking the Euler characteristic of both sides that

$$1 = \sum_{\tau_p \in \mathcal{F}_\infty /\ i(\tau_p) \le k} (-1)^{i(\tau_p)}.$$

Such an equality contradicts the assumption of Theorem 1.1.

Now, for generic K, it follows from the Sard–Smale theorem that all solutions of 1.4 are non-degenerate solutions, see [37]. We apply the Euler–Poincaré characteristic argument, we derive from (4.2) that

$$1 = \sum_{\tau_p \in \mathcal{F}_\infty /\ i(\tau_p) \le k} (-1)^{i(\tau_p)} + \sum_{\omega < \theta(X_{k_0}^\infty), i(\omega) \le k+1} (-1)^{i(\omega)}.$$

It follows then that

$$\left| 1 - \sum_{\tau_p \in \mathcal{F}_\infty / \ i(\tau_p) \leq k} (-1)^{i(\tau_p)} \right| \leq \ \sharp \mathcal{N}_{k+1},$$

where $\mathcal{N}_{k+1}$ denotes the set of solutions of (1.4) having their Morse indices $\leq k + 1$. This conclude the proof of Theorem 1.1.

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